Weak annihilation in the rare radiative $B \to \rho\gamma$ decay

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The amplitude of the $B \to \rho\gamma$ decay induced by the flavour-changing neutral currents contains the penguin contribution and the weak-annihilation contribution generated by the 4-quark operators in the effective Hamiltonian. The penguin contribution is known quite well. We analyze the weak-annihilation which is suppressed by the heavy-quark mass compared to the penguin contribution.

In the factorization approximation, the weak annihilation amplitude is represented in terms of the leptonic decay constants and the meson-photon matrix elements of the weak currents. The latter contain the $B\gamma$, $\rho\gamma$ transition form factors and contact terms determined by the equations of motion. We calculate the $B\gamma$ and $\rho\gamma$ form factors within the relativistic dispersion approach and obtain numerical estimates for the weak annihilation amplitude.

I. INTRODUCTION

The investigation of rare semileptonic B decays induced by the flavour-changing neutral current transitions $b \to s$ and $b \to d$ represents an important test of the Standard Model or its extentions. Rare decays are forbidden at the tree level in the Standard Model and occur through loop diagrams. Thus they provide the possibility to probe the structure of the electroweak sector at large mass scales from contributions of virtual particles in the loop diagrams. Interesting information about the structure of the theory is contained in the Wilson coefficents in the effective Hamiltonian which describes the $b \to s, d$ transition at low energies. These Wilson coefficients take different values in different theories with testable consequences in rare B decays.

Among rare B decays the radiative decays $b \to s\gamma$ and $b \to d\gamma$ have the largest probabilites. The $b \to s\gamma$ transitions are CKM favoured and have larger branching ratios than the $b \to d\gamma$ transitions. The $b \to s\gamma$ transition has been observed by CLEO in the exclusive channel $B \to K^*\gamma$ in 1993 and measured inclusively in 1995. The $B \to \rho\gamma$ decay will be extensively studied by BaBar and BELLE.

The main uncertainty in the theoretical analysis of B decays is connected with long-distance QCD effects arising from the presence of hadrons in initial and final states. In inclusive decays these effects are under better control, however from inclusive measurements it is more difficult to obtain precise results.

The decay amplitude contains two different contributions: one arising from the electromagnetic penguin operator and another from the 4-fermion operators in the effective Hamiltonian. One of the effects generated by the 4-fermion operators is the weak annihilation (WA). Further details about short- and long-distance effects in the radiative decays can be found in recent publications [1–4] and references therein.

In the $B \to K^*\gamma$ decay the weak annihilation is negligible compared to the penguin effect: it is suppressed by two powers of the small parameter $\lambda \simeq 0.2$ of the Cabbibo-Kabayashi-Maskawa (CKM) matrix. In $B \to \rho\gamma$ both effects have the same order in λ and must be taken into account.

The penguin contribution has been analyzed in several ways and is known quite well. On the other hand, the WA in $B \to \rho\gamma$ has been studied in less detail: the relevant form factors were analyzed within sum rules [5,6] and perturbative QCD [7]. However, some contributions to these form factors were neglected. These contributions may be relevant if precise measurements become available.

The aim of this paper is to analyze the weak annihilation for the $B^- \to \rho^-\gamma$ decay more closely.

In the factorization approximation, the weak-annihilation amplitude can be represented as the product of meson leptonic decay constants and matrix elements of the weak current between meson and photon. The latter contain the meson-photon transition form factors and contact terms which are determined by the equations of motion. The photon can be emitted from the loop containing the b quark which is described by $B\gamma$ transition form factors. It can also be emitted from the loop containing only light quarks described by the $\rho\gamma$ transition form factors (Fig 1).

We calculate the $B\gamma$ form factors within the relativistic dispersion approach which expresses these form factors in terms of the B meson wave function. We demonstrate that the form factors calculated by the dispersion approach behave in the limit $m_b \to \infty$ in agreement with perturbative QCD. The B meson wave function was previously tested in the $B \to$ light meson weak decays and is known quite well, allowing us to provide reliable numerical estimates for the $B\gamma$ form factors.

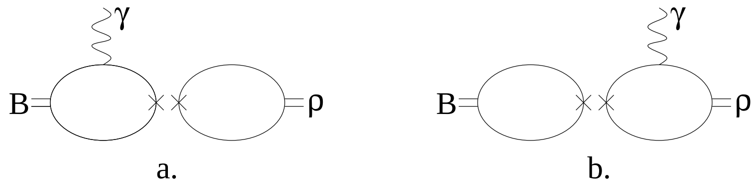

FIG. 1. Diagrams describing the weak annihilation process for $B \to \rho\gamma$ in the factorization approximation: (a) The photon is emitted from the loop containing the b quark, (b) The photon is emitted from the loop containing only light quarks.

The $\rho\gamma$ transition form factor is related to the divergence of the vector and axial-vector currents. In the case of the axial-vector current it is proportional to the light-quark masses if the classical equation of motion is applied. Because the quark momenta in the loop are high, these masses have to be identified with current quark masses. For this reason the corresponding $\rho\gamma$ form factor was neglected in previous analyses [1,5]. We find however that this argument is not correct: this form factor remains finite in the limit of vanishing light quark mass $m \to 0$ and behaves like $\sim M_\rho f_\rho / M_B^2$ which means the violation of the classical equation of motion. We present the result for the $\rho\gamma$ transition form factor but leave a detailed discussion of the anomaly appearing for the matrix element $\langle \rho\gamma | \partial_\nu A_\nu | 0 \rangle$ for a special publication.

Finally, we provide numerical estimates of the weak-annihilation amplitude taking into account the $B\gamma$, $\rho\gamma$ transition form factors, and the contact term contributions.

In Section II the effective Hamiltonian for the $b \to d$ transition and the general structure of the amplitude are presented. In Section III we discuss the photon emission from the B meson loop and obtain the $B\gamma$ transition form factors within the dispersion approach. Section IV contains results for the $\rho\gamma$ transition form factors. In Section V the numerical estimates are given. The concluding Section summarises our results.

II. THE EFFECTIVE HAMILTONIAN, THE AMPLITUDE AND THE DECAY RATE

The amplitude of the weak radiative $B \to \rho$ transition is given by the matrix element of the effective Hamiltonian for the $b \to d$ transition

$$A(B \to \rho\gamma) = \langle \gamma(q_1)\rho(q_2) | H_{\rm eff}(b \to d) | B(p) \rangle, \tag{1}$$

where p is the B momentum, q_2 is the ρ momentum, and q_1 is the photon momentum, $p = q_1 + q_2$, $q_1^2 = 0$, $q_2^2 = M_\rho^2$, $p^2 = M_B^2$. The effective weak Hamiltonian has the structure [8]:

$$H_{\rm eff}(b \to d) = \frac{G_F}{\sqrt{2}}\, \xi_t C_{7\gamma}(\mu) \mathcal{O}_{7\gamma} - \frac{G_F}{\sqrt{2}} \xi_u \left(C_1(\mu)\mathcal{O}_1 + C_2(\mu)\mathcal{O}_2 \right), \tag{2}$$

where only operators relevant for the penguin and weak ahnnihilation effects are listed. G_F is the Fermi constant, $\xi_q = V_{qd}^* V_{qb}$, C_i's are the Wilson coefficients and $\mathcal{O}_i$'s are the basis operators

$$\mathcal{O}_{7\gamma} = \frac{e}{8\pi^2}\, \bar{d}_\alpha \sigma_{\mu\nu} m_b(\mu) \left(1 + \gamma_5\right)\, b_\alpha\, F_{\mu\nu}, \tag{3}$$

$$\begin{aligned} \mathcal{O}_1 &= \bar{d}_\alpha \gamma_\nu (1 - \gamma_5) u_\alpha\ \bar{u}_\beta \gamma_\nu (1 - \gamma_5) b_\beta, \\ \mathcal{O}_2 &= \bar{d}_\alpha \gamma_\nu (1 - \gamma_5) u_\beta\ \bar{u}_\beta \gamma_\nu (1 - \gamma_5) b_\alpha, \end{aligned} \tag{4}$$

with the following notation: $e = \sqrt{4\pi\alpha_{\rm em}}$, $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, $\sigma_{\mu\nu} = i\left[\gamma_\mu, \gamma_\nu\right]/2$, $\epsilon^{0123} = -1$ and ${\rm Sp}\left(\gamma^5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\right) = 4i\epsilon^{\mu\nu\alpha\beta}$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

The amplitude can be parametrized as follows

$$A(B^- \to \rho^-\gamma) = \frac{eG_F}{\sqrt{2}} \left[\epsilon_{q_1 \epsilon_1^* q_2 \epsilon_2^*} F_{\rm PC} + i\epsilon_2^{*\nu} \epsilon_1^{*\mu} \left(g_{\nu\mu}\, pq_1 - p_\mu q_{1\nu} \right) F_{\rm PV} \right], \tag{5}$$

where $F_{\rm PC}$ and $F_{\rm PV}$ are the parity-conserving and parity-violating invariant amplitudes, respectively. $\epsilon_2(\epsilon_1)$ is the ρ-meson (photon) polarization vector. We use the short-hand notation $\epsilon_{abcd} = \epsilon_{\alpha\beta\mu\nu} a^\alpha b^\beta c^\mu d^\nu$ for any 4-vectors a, b, c, d.

For the decay rate one finds

$$\Gamma(B^- \to \rho^-\gamma) = \frac{G_F^2\, \alpha_{em}}{16} M_B^3 \left(1 - M_\rho^2/M_B^2\right)^3 \left(|F_{\rm PC}|^2 + |F_{\rm PV}|^2 \right). \tag{6}$$

A. The penguin amplitude

The main contribution to the amplitude is given by the electromagnetic penguin operator $O_{7\gamma}$:

$$A_{\rm peng}(B\to\rho\gamma) = -\frac{eG_F}{\sqrt{2}}\xi_t C_7 \frac{m_b}{2\pi^2}T_1(0)\left(\epsilon_{q_1\epsilon_1^* q_2\epsilon_2^*} + i\epsilon_2^{*\nu}\epsilon_1^{*\mu}(g_{\nu\mu}\,pq_1 - p_\mu q_{1\nu})\right), \tag{7}$$

where T_1 is the form factor of the $B\to\rho$ transition through the tensor current [9–11]. The corresponding contribution to the invariant amplitudes is therefore

$$F_{\rm PC}^{\rm peng} = F_{\rm PV}^{\rm peng} = -\xi_t C_7 \frac{m_b}{2\pi^2}T_1(0). \tag{8}$$

B. The weak annihilation amplitude

The radiative $B\to\rho\gamma$ transition also receives contribution from the 4-fermion operators $\mathcal{O}_1$ and $\mathcal{O}_2$. For the charged $B^-\to\rho^-(q_2)\gamma(q_1)$ transition the corresponding amplitude reads

$$A_{\rm WA}(B^-\to\rho^-\gamma) = -\frac{G_F}{\sqrt{2}}\xi_u\langle\rho(q_2)\gamma(q_1)|\bar d\gamma_\nu(1-\gamma_5)u\cdot\bar u\gamma_\nu(1-\gamma_5)b|B(p)\rangle, \tag{9}$$

In what follows we suppress the lable WA. Neglecting the nonfactorizable soft-gluon exchanges, i.e. assuming vacuum saturation, the complicated matrix element in Eq. (9) is reduced to simpler quantities - the meson-photon matrix elements of the bilinear quark currents and the meson decay constants. The latter are defined as usual

$$\begin{aligned}\langle\rho(q_2)|\bar d\gamma_\nu u|0\rangle &= \epsilon_{2\nu}^* M_\rho f_\rho, \qquad f_\rho>0,\\ \langle 0|\bar u\gamma_\nu\gamma_5 b|B(p)\rangle &= ip_\nu f_B, \qquad f_B>0.\end{aligned} \tag{10}$$

It is convenient to isolate the parity-conserving contribution which emerges from the product of the two equal-parity currents, and the parity-violating contribution which emerges from the product of the two opposite-parity currents.

1. The parity-violating amplitude

The parity-violating amplitude has the form

$$A_{\rm PV}(B\to\rho\gamma) = \frac{G_F}{\sqrt{2}}\xi_u a_1\left\{\langle\rho\gamma|\bar d\gamma_\nu u|0\rangle\langle 0|\bar u\gamma_\nu\gamma_5 b|B\rangle + \langle\rho|\bar d\gamma_\nu u|0\rangle\langle\gamma|\bar u\gamma_\nu\gamma_5 b|B\rangle\right\}. \tag{11}$$

Here a_1 is an effective Wilson coefficient, which we take as $a_1 = C_1 + C_2/N_c$ at the scale ~ 5 GeV.

The first term is a contact term which can be calculated using the equations of motion for the quark fields [1]. Setting $m_u = m_d$, we find

$$\langle\rho\gamma|\bar d\gamma_\nu u|0\rangle\langle 0|\bar u\gamma_\nu\gamma_5 b|B\rangle = ip_\nu f_B\langle\rho\gamma|\bar d\gamma_\nu u|0\rangle = f_B\langle\rho\gamma|\partial_\nu(\bar d\gamma_\nu u)|0\rangle = ie\epsilon_2^{*\nu}\epsilon_1^{*\mu}\, g_{\mu\nu}M_\rho f_\rho f_B. \tag{12}$$

The $B\to\gamma$ amplitude in the second term of (11) can be parametrized as follows

$$\langle\gamma(q_1)|\bar u\gamma_\nu\gamma_5 b|B(p)\rangle = ie\epsilon_1^{*\mu}\left[(g_{\nu\mu}\,pq_1 - p_\mu q_{1\nu})\frac{2F_A}{M_B} - p_\mu q_{1\nu}\frac{2f_B}{M_B^2-M_\rho^2}\right]. \tag{13}$$

It contains the form factor $F_A(q_2^2 = M_\rho^2)$, and the contact term proportional to f_B (the derivation of this relation is given in the Appendix).

Summing the contributions of the photon emission from the B-meson loop and the ρ-meson loop gives the amplitude $A_{\rm PV}$ which can be represented in the form $A_{\rm PV} = \epsilon_1^{*\mu}T_\mu$ with $q_1^\mu T_\mu = 0$ as required by gauge invariance. Thus, the weak-annihilation contribution to the form factor $F_{\rm PV}$ for the $B^-\to\rho^-\gamma$ decay is[1]

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[1]We note that for the $B^+\to\rho^+\gamma$ the term proportional to f_B changes sign (see Appendix), and also $F_A^{B^+} = -F_A^{B^-}$ as can be obtained from charge conjugation of the amplitude $A_{\rm PV}$. For the $B^0\to\rho^0\gamma$ decay the term $\sim f_B$ is absent.

$$F_{\rm PV}^{\rm WA} = \xi_u a_1 f_\rho M_\rho \left[\frac{2F_A}{M_B} + \frac{2f_B}{M_B^2 - M_\rho^2}\right]. \tag{14}$$

The two contact terms which are present in the amplitudes of the photon emission from the B meson loop and from the ρ-meson loop do not cancel each other (we disagree here with the claim of Ref. [6]) but lead to a nonvanishing contribution proportional to f_B.

C. The parity-conserving amplitude

This amplitude reads

$$A_{\rm PC}(B \to \rho\gamma) = -\frac{G_F}{\sqrt{2}}\xi_u a_1 \left\{\langle\rho|\bar{d}\gamma_\nu u|0\rangle\langle\gamma|\bar{u}\gamma_\nu b|B\rangle + \langle\gamma\rho|\bar{d}\gamma_\nu\gamma_5 u|0\rangle\langle 0|\bar{u}\gamma_\nu\gamma_5 b|B\rangle\right\}. \tag{15}$$

The $B \to \gamma$ amplitude from the first term in the brackets contains the form factor $F_V(q_2^2 = M_\rho^2)$:

$$\langle\gamma(q_1)|\bar{u}\gamma_\nu b|B(p)\rangle = e\,\epsilon_{\nu\mu p q_1}\epsilon_1^{*\mu}\,\frac{2F_V}{M_B}. \tag{16}$$

The second term in (15) can be reduced to the divergence of the axial-vector current and contains another form factor, G_V: namely,

$$\langle 0|\bar{u}\gamma_\nu\gamma_5 b|B\rangle\langle\gamma\rho|\bar{d}\gamma_\nu\gamma_5 u|0\rangle = f_B\langle\gamma\rho|\partial_\nu\bar{d}\gamma_\nu\gamma_5 u|0\rangle = e\,f_B G_V\epsilon_{q_1\epsilon_1^* q_2\epsilon_2^*}. \tag{17}$$

Thus, the weak annihilation contribution to $F_{\rm PC}$ reads

$$F_{\rm PC}^{\rm WA} = \xi_u a_1 M_\rho f_\rho \left[\frac{2F_V}{M_B} - \frac{f_B\,G_V}{M_\rho f_\rho}\right]. \tag{18}$$

Summing up, within the factorization approximation the weak annihilation amplitude can be expressed in terms of the three form factors F_A, F_V, and G_V.

III. THE FORM FACTORS F_A AND F_V

In this section we derive the formulas for the form factors $F_{A,V}$ within the dispersion approach to the transition form factors. This approach has been formulated in detail in [10] and applied to the weak decays of heavy mesons in [11]. Recall that the form factors $F_{A,V}$ describe the transition of the B-meson to the photon with the momentum q_1, $q_1^2 = 0$, induced by the axial-vector (vector) current with the momentum transfer q_2, $q_2^2 = M_\rho^2$. For technical reasons, it is convenient to treat the form factor $F_{A(V)}$ as describing the amplitude of the photon-induced transition of the B-meson into a $b\bar{u}$ axial-vector (vector) virtual particle with the corresponding factor $1/(s - q_2^2)$ in the dispersion integral. Then we can directly apply the equations obtained in [10] for the meson-meson transition form factors.

A. The form factor F_A

The form factor F_A is given by the diagrams of Fig 2. Fig 2a shows $F_A^{(b)}$, the contribution to the form factor of the process when the b quark interacts with the photon; Fig 2b describes the contribution of the process when the quark u interacts while b remains a spectator. One has

$$F_A = F_A^{(b)} + F_A^{(u)}. \tag{19}$$

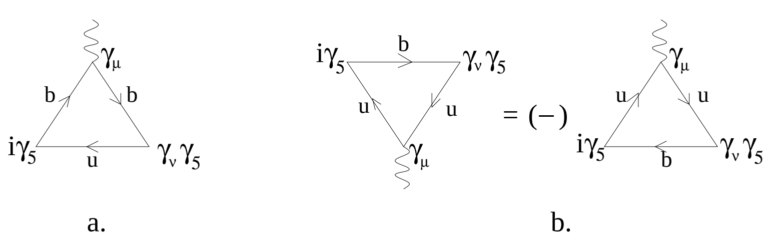

FIG. 2. Diagrams for the form factor F_A: a) $F_A^{(b)}$, b) $F_A^{(u)}$.

The B^- meson is described by the vertex $\bar{b}(k_b)\, i\gamma_5 u(k_u)\, G(s)/\sqrt{N_c}$, with $G(s) = \phi_B(s)(s - M_B^2)$. The B-meson wave function ϕ_B is normalized according to the relation [10]

$$\frac{1}{8\pi^2} \int\limits_{(m_b+m_u)^2}^{\infty} ds\phi_B^2(s)\left(s - (m_b - m_u)^2\right) \frac{\lambda^{1/2}(s, m_b^2, m_u^2)}{s} = 1. \tag{20}$$

Here $\lambda(a, b, c) = (a - b - c)^2 - 4bc$ is the triangle function.

It is convenient to change the direction of the quark line in the loop diagram of Fig 2b. This is done by performing the charge conjugation of the matrix element and leads to a sign change for the $\gamma_\nu\gamma_5$ vertex.

Now both diagrams in Fig 2 a,b are reduced to the diagram of Fig 3 which defines the form factor $F_A^{(1)}(m_1, m_2)$: Setting $m_1 = m_b$, $m_2 = m_u$ gives $F_A^{(b)}$: $F_A^{(b)} = Q_b F_A^{(1)}(m_b, m_u)$. Similarly, setting $m_1 = m_u$, $m_2 = m_b$ gives $F_A^{(u)}$, $F_A^{(u)} = -Q_u F_A^{(1)}(m_u, m_b)$.

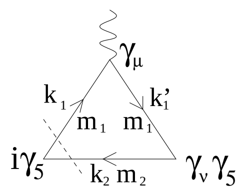

FIG. 3. The triangle diagram for $F_A^{(1)}(m_1, m_2)$. The cut corresponds to calculating the imaginary part in the variable p^2.

For the diagram of Fig 3 (quark 1 emits the photon, quark 2 is the spectator) the trace reads

$$-\,\mathrm{Sp}\; i\gamma_5(m_2 - \hat{k}_2)\gamma_\nu\gamma_5(m_1 + \hat{k}_1')\gamma_\mu(m_1 + \hat{k}_1) = 4i(k_1 + k_1')_\mu(m_1 k_2 + m_2 k_1)_\nu + 4i(g_{\mu\nu}q_\alpha - g_{\mu\alpha}q_\nu)(m_1 k_2 + m_2 k_1)_\alpha.$$

The spectral density of the form factor $F_A^{(1)}(m_1, m_2)$ in the variable p^2, $p = k_1 + k_2$, is the coefficient of the structure $g_{\mu\nu}$ obtained after the integration of the trace over the quark phase space. Performing necessary calculations, we arrive at the following single dispersion integral

$$\begin{aligned}\frac{2}{M_B}F_A^{(1)} = \frac{\sqrt{N_c}}{4\pi^2}\int\limits_{(m_b+m_u)^2}^{\infty} \frac{ds\;\phi_B(s)}{(s - M_\rho^2)}\Bigg\{&\left(m_1 \log\left(\frac{s + m_1^2 - m_2^2 + \lambda^{1/2}(s, m_1^2, m_2^2)}{s + m_1^2 - m_2^2 - \lambda^{1/2}(s, m_1^2, m_2^2)}\right) + (m_2 - m_1)\frac{\lambda^{1/2}(s, m_b^2, m_u^2)}{s}\right)\\ &+ \frac{1}{pq_1}\left(\frac{\lambda^{1/2}(s, m_b^2, m_u^2)}{2s} - m_1^2 \log\left(\frac{s + m_1^2 - m_2^2 + \lambda^{1/2}(s, m_1^2, m_2^2)}{s + m_1^2 - m_2^2 - \lambda^{1/2}(s, m_1^2, m_2^2)}\right)\right)\Bigg\}.\end{aligned} \tag{21}$$

For $q_1^2 = 0$ one has $pq_1 = (M_B^2 - M_\rho^2)/2$.

Now, let us analyze the behaviour of the form factor in the limit $m_b \to \infty$. To this end it is convenient to rewrite the spectral representation (21) in terms of the light-cone variables as follows (see [12] for details)

$$\frac{2}{M_B}F_A^{(1)}(m_1, m_2) = \frac{\sqrt{N_c}}{4\pi^2}\int \frac{dx_1 dx_2 dk_\perp^2}{x_1^2 x_2}\delta(1 - x_1 - x_2)\frac{\phi_B(s)}{s - M_\rho^2}\left\{m_1 x_2 + m_2 x_1 - (m_1 - m_2)k_\perp^2/pq_1\right\}. \tag{22}$$

Here x_i is the fraction of the B-meson light-cone momentum carried by the quark i, and $s = m_1^2/x_1 + m_2^2/x_2 + k_\perp^2/x_1 x_2$. For the form factors $F_A^{(u)}$ and $F_A^{(b)}$ one obtains

$$\begin{aligned}\frac{2}{M_B}F_A^{(u)} &= -Q_u\frac{\sqrt{N_c}}{4\pi^2}\int \frac{dx dk_\perp^2}{x_u^2 x_b}\frac{\phi_B(s)}{s - M_\rho^2}\left\{m_u\, x_b + m_b\, x_u + \frac{2(m_b - m_u)k_\perp^2}{M_B^2 - M_\rho^2}\right\},\\ \frac{2}{M_B}F_A^{(b)} &= \quad Q_b\frac{\sqrt{N_c}}{4\pi^2}\int \frac{dx dk_\perp^2}{x_b^2 x_u}\frac{\phi_B(s)}{s - M_\rho^2}\left\{m_b\, x_u + m_u\, x_b + \frac{2(m_u - m_b)k_\perp^2}{M_B^2 - M_\rho^2}\right\},\end{aligned}$$

with

$$s = \frac{m_b^2}{x_b} + \frac{m_u^2}{x_u} + \frac{k_\perp^2}{x_u x_b}. \tag{23}$$

Let us recall that the B-meson decay constant has the following representation in terms of the wave function [10]:

$$f_B = \frac{\sqrt{N_c}}{4\pi^2} \int \frac{dx dk_\perp^2}{x_u x_b} \frac{\phi_B(s)}{s - M_\rho^2} \{m_u\, x_b + m_b\, x_u\}\,. \tag{24}$$

Due to the wave function $\phi_B(s)$, the integral in the heavy quark limit is dominated by the region $x_u = \bar\Lambda/m_b$, $x_b = 1 - \bar\Lambda/m_b$, where $\bar\Lambda$ is a constant of order $M_B - m_b$. This leads to the following expansion of the form factors in the $1/m_b$ series

$$\begin{aligned} \frac{2}{M_B} F_A^{(u)} &= -Q_u \frac{f_b}{\bar\Lambda m_b} + ... \\ \frac{2}{M_B} F_A^{(b)} &= \quad Q_b \frac{f_b}{m_b^2} + ... \end{aligned} \tag{25}$$

Clearly, the dominant contribution in the heavy quark limit comes from the process when the light quark emits the photon, whereas the emission of the photon from the heavy quark gives only a $1/m_b$ correction. The expressions (25) for the form factor $F_A^{(u)}$ agrees with the result of Ref. [7], while we have found a different sign for $F_A^{(b)}$.

B. The form factor F_V

The consideration of the form factor F_V is very similar to the form factor F_A. F_V is determined by the two diagrams shown in Fig 4: Fig 4a gives $F_V^{(b)}$, the contribution of the process when the b quark interacts with the photon; Fig 4b describes the contribution of the process when the quark u interacts. One has

$$F_V = F_V^{(b)} + F_V^{(u)}. \tag{26}$$

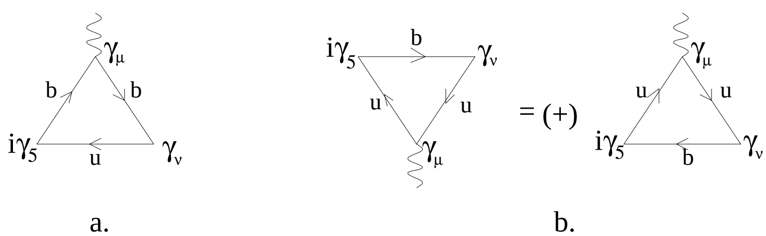

FIG. 4. Diagrams for the form factor F_V: a) $F_V^{(b)}$, b) $F_V^{(u)}$.

It is again convenient to change the direction of the quark line in the loop diagram of Fig 4b describing $F_V^{(u)}$ by performing the charge conjugation of the matrix element. For the vector current γ_ν in the vertex the sign does not change (in contrast to the $\gamma_\nu\gamma_5$ case considered above).

Then both diagrams in Fig 4 a, b are reduced to the diagram of Fig 5 which gives the form factor $F_V^{(1)}(m_1, m_2)$: Setting $m_1 = m_b$, $m_2 = m_u$ gives $F_V^{(b)}$: $F_V^{(b)} = Q_b F_V^{(1)}(m_b, m_u)$; Setting $m_1 = m_u$, $m_2 = m_b$ gives $F_V^{(u)}$, $F_V^{(u)} = Q_u F_V^{(1)}(m_u, m_b)$.

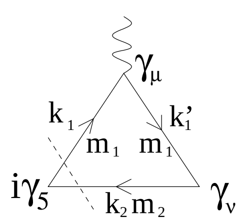

FIG. 5. The triangle diagram for $F_V^{(1)}(m_1, m_2)$. The cut corresponds to calculating the imaginary part in the variable p^2.

The trace corresponding to the diagram of Fig 4 (1 - active quark, 2 - spectator) reads

$$-\,\mathrm{Sp}\ i\gamma_5(m_2-\hat{k}_2)\gamma_\nu(m_1+\hat{k}'_1)\gamma_\mu(m_1+\hat{k}_1)=-4\epsilon_{\nu\mu\alpha q_1}(m_1k_2+m_2k_1)_\alpha.$$

The spectral representation for the form factor takes the form

$$\frac{2}{M_B}F_V^{(1)}=\frac{\sqrt{N_c}}{4\pi^2}\int\limits_{(m_b+m_u)^2}^{\infty}\frac{ds\phi_B(s)}{(s-M_\rho^2)}\left\{(m_2-m_1)\frac{\lambda^{1/2}(s,m_b^2,m_u^2)}{s}+m_1\log\left(\frac{s+m_1^2-m_2^2+\lambda^{1/2}(s,m_1^2,m_2^2)}{s+m_1^2-m_2^2-\lambda^{1/2}(s,m_1^2,m_2^2)}\right)\right\}.$$

To analyze the heavy quark limit $m_b\to\infty$ we again represent the form factor in terms of the light-cone variables

$$\frac{2}{M_B}F_V^{(1)}=-\frac{\sqrt{N_c}}{4\pi^2}\int\frac{dx_1dx_2dk_\perp^2}{x_1^2x_2}\delta(1-x_1-x_2)\frac{\phi_B(s)}{s-M_\rho^2}\left(m_1x_2+m_2x_1\right).\tag{27}$$

For $F_V^{(u)}$ and $F_V^{(b)}$ the corresponding expressions read

$$\frac{2}{M_B}F_V^{(u)}=-Q_u\frac{\sqrt{N_c}}{4\pi^2}\int\frac{dxdk_\perp^2}{x_u^2x_b}\frac{\phi_B(s)}{s-M_\rho^2}\left\{m_u\,x_b+m_b\,x_u\right\},$$
$$\frac{2}{M_B}F_V^{(b)}=-Q_b\frac{\sqrt{N_c}}{4\pi^2}\int\frac{dxdk_\perp^2}{x_b^2x_u}\frac{\phi_B(s)}{s-M_\rho^2}\left\{m_b\,x_u+m_u\,x_b\right\},$$

By the same procedure as used for F_A, we obtain in the limit $m_b\to\infty$

$$\frac{2}{M_B}F_V^{(u)}=-Q_u\frac{f_b}{\bar{\Lambda}m_b}+\dots$$
$$\frac{2}{M_B}F_V^{(b)}=-Q_b\frac{f_b}{m_b^2}+\dots\tag{28}$$

The dominant contribution in the heavy quark limit again comes from the process when the light quark emits the photon. Now both form factors $F_V^{(u)}$ and $F_V^{(b)}$ in (28) perfectly agree with the expansions obtained in [7].

As seen from Eqs. (25) and (28), one finds $F_A=F_V$ in the heavy quark limit, in agreement with the large-energy effective theory [13].

IV. THE FORM FACTOR G_V

The form factor G_V is determined by the divergence of the matrix element of the charged current between the vacuum and the $\rho^-\gamma$ state,

$$ip_\nu\langle\gamma(q_1)\rho^-(q_2)|\bar{d}\gamma_\nu\gamma_5u|0\rangle=eG_V\epsilon_{q_1\epsilon_1^*q_2\epsilon_2^*}.\tag{29}$$

The corresponding diagrams are shown in Fig 6.

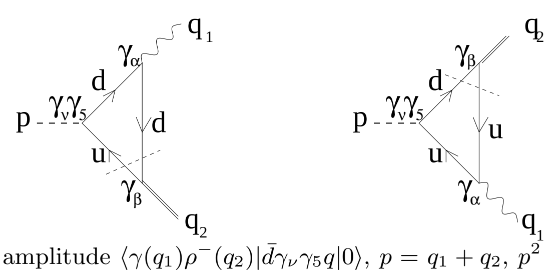

FIG. 6. Diagrams describing the amplitude $\langle\gamma(q_1)\rho^-(q_2)|\bar{d}\gamma_\nu\gamma_5q|0\rangle$, $p=q_1+q_2$, $p^2=M_B^2$. The diagram (a) is multiplied by the d-quark charge, and the the diagram (b) is multiplied by the u-quark charge. The cut corresponds to the calculation of the imaginary part in the variable q_2^2.

If the classical equations of motion are applied, the form factor G_V is proportional to the light-quark masses. This is the reason why this form factor was neglected in previous analyses [1,5]. However, a proper calculation shows that this argument is not correct: in fact the classical equations of motions do not hold and the divergence contains the anomaly.

The anomalous behavior of the divergence of the axial-vector current in the chiral limit is a well-known phenomenon discovered in the two-photon amplitude $\langle\gamma\gamma|\partial_\nu\bar{q}\gamma_\nu\gamma_5 q|0\rangle$ [14]. A very clear way to demonstrate the anomaly is to start with the matrix element of the axial vector current and to calculate the spectral representations for the relevant form factors. The anomaly is then obtained by performing the divergence at the final stage of the calculation [15]. A similar treatment applied to the matrix element $\langle\gamma\rho^-|\bar{d}\gamma_\nu\gamma_5 u|0\rangle$ leads to the form factor G_V which does not vanish in the limit $m\to 0$. We will present a detailed discussion of the anomaly in radiative B decays in a separate publication [16]. Here we only quote the final result: considering the spectral representation in the variable q_2^2 and introducing the appropriate ρ-meson radial wave function $\psi_\rho(s)$, one finds the following expression for the form factor G_V

$$G_V=\sqrt{N_c}(Q_u+Q_d)\left[-\frac{M_B^2}{4\pi^2}\int\frac{ds\,\psi_\rho(s)\,(s-M_\rho^2)}{(s-M_B^2-i0)^2}\right]. \tag{30}$$

$\psi_\rho(s)$ is normalized according to the relation (for massless quarks)

$$\frac{1}{8\pi^2}\int\limits_0^\infty ds\; s\;|\psi_\rho(s)|^2=1. \tag{31}$$

Clearly, G_V is finite for $m=0$ which means a violation of the classical equations of motion for the axial-vector current. Eq. (30) describes the anomaly which takes place for the $\gamma\rho$ final state as well as for the $\gamma\gamma$ one. There is however an important difference between the two cases: For the $\gamma\gamma$ final state the divergence remains finite in the limit $m\to 0$ and $p^2=M_B^2\to\infty$. For the $\rho\gamma$ final state the divergence is finite for $m\to 0$ but decreases as $1/p^2$ for $p^2\to\infty$. It is convenient to introduce the parameter κ such that

$$G_V=(Q_u+Q_d)\,\kappa\frac{M_\rho f_\rho}{M_B^2}, \tag{32}$$

with κ staying finite for $m=0$ and $M_B\to\infty$.

V. NUMERICAL ESTIMATES

Let us write once more the penguin and the weak annihilation contributions to the $B^-\to\rho^-\gamma$ amplitude:

$$\begin{aligned}
F_{\rm PV}^{\rm peng}&=F_{\rm PC}^{\rm peng}=-\xi_t C_7\frac{m_b}{2\pi^2}T_1(0),\\
F_{\rm PV}^{\rm WA}&=\xi_u a_1 M_\rho f_\rho\,\left[\frac{2F_A}{M_B}+\frac{2f_B}{M_B^2-M_\rho^2}\right].\\
F_{\rm PC}^{\rm WA}&=\xi_u a_1 M_\rho f_\rho\,\left[\frac{2F_V}{M_B}-\frac{f_B G_V}{M_\rho f_\rho}\right].
\end{aligned} \tag{33}$$

The scaling behavior of the form factors in the limit $M_B\to\infty$ reads

$$T_1(0)\sim M_B^{-3/2}\ [17],\quad F_{A,B}\sim M_B^{-1/2},\quad G_V\sim M_B^{-2} \tag{34}$$

such that

$$F^{\rm peng}\sim M_B^{-1/2},\quad F_{\rm PV,PC}^{\rm WA}\sim M_B^{-3/2}. \tag{35}$$

The terms proportional to f_B ($f_B\sim 1/\sqrt{M_B}$) are $1/M_B$-suppressed compared with the terms containing $F_{A,V}$. As we see below numerically this leads to a suppression by a factor of $4-5$.

We now proceed to numerical estimates for the B-meson decay. The scale-dependent Wilson coefficients $C_i(\mu)$ and $a_1(\mu)$ take the following values at the renormalization scale $\mu\simeq 5$ GeV [8]:

$$C_1=1.1,\qquad C_2=-0.241,\qquad C_{7\gamma}=-0.312,\qquad a_1=C_1+C_2/N_c\simeq 1.02. \tag{36}$$

The penguin form factor was previously calculated within the dispersion approach with the result $T_1^{B^-\to\rho^-}(0) = 0.27 \pm 0.3$ [11]. Using the same parameters and the B meson wave function as in [11] we obtain the form factors $F_{A,B}$ shown in Table I. Our result for the form factor F_V is in good agreement with the estimates from other approaches. The form factor F_A agrees well with the constraints from perturbative QCD and turns out to be considerably larger than the corresponding sum rule estimate.

The value of the G_V is very sensitive to the details of the ρ meson wave function ψ_ρ. The reason for that is the presence of the term $(s - M_\rho^2)$ in the integrand in Eq. (30) which changes sign in the integration region. Assuming $\psi_\rho(s) \simeq \beta^2/(\beta^2+s)^2$ and setting $\beta = 0.8$ GeV, which gives a good description of the ρ meson radius, leads to $\kappa = -1.8$. However, this value has a large uncertainty. Conservatively, we take $|\kappa| < 2.0$ and use this result for further estimates. The form factor G_V does not contribute more than a few % to the full amplitude, but can sizeably correct the weak-annihilation part.

Using the obtained form factors and the decay constants $f_B = 0.18$ GeV, $f_\rho = 0.21$ GeV we arrive at the following estimates (we show contributions of different terms in Eq. (33) separately):

$$\begin{aligned} F^{\rm peng} &= \xi_t\ 20\ \text{MeV}, \\ F^{\rm WA}_{\rm PV} &= \xi_u\ \{-7.8\ (\text{cont. of } F_A) +\ 2.2\ (\text{cont. of } f_B)\}\ \text{MeV} = -5.6\ \xi_u\ \text{MeV} \\ F^{\rm WA}_{\rm PC} &= \xi_u\ \{-6.0\ (\text{cont. of } F_V) \pm\ 0.7(\text{cont. of G}_\text{V})\}\ \text{MeV} = -(6.0 \pm 0.7)\ \xi_u\ \text{MeV}. \end{aligned} \tag{37}$$

Taking into account errors in the form factors, for the ratios of the weak-annihilation to the penguin amplitudes we find

$$\begin{aligned} \text{parity} - \text{violating}: &\qquad F^{\rm WA}_{\rm PV}/F^{\rm peng}_{\rm PV} \simeq -(0.28 \pm 0.025)\ \xi_u/\xi_t, \\ \text{parity} - \text{conserving}: &\qquad F^{\rm WA}_{\rm PC}/F^{\rm peng}_{\rm PC} \simeq -(0.3 \pm 0.05)\ \xi_u/\xi_t. \end{aligned} \tag{38}$$

$|\xi_u/\xi_t| \simeq 0.4$ in the Standard Model.

Our results for $F^{\rm peng}$ and $F^{\rm WA}$ agree with the sum rules [6] which reported the ratio $F^{\rm WA}/F^{\rm peng} \simeq -0.3\ \xi_u/\xi_t$. We would like to notice, however, that the anatomy of $F^{\rm WA}$ in our analysis is different. For instance, we have found F_A considerably larger than the sum rule result. But after including the contact term $\sim f_B$ which was not taken into account in [6], we have come to $F^{\rm WA}_{\rm PV}$ close to the sum rule estimate.

The calculated form factors and the ratios of the weak-annihilation to the penguin amplitudes is one of the necessary ingredients for the calculation of the branching ratios of the $B \to \rho\gamma$ decays, Isospin and CP Asymmetries. In addition to the above ratios, these quantities contain the phase induced by the strong interactions and the CP-violating phase of the CKM matrix (see [3,4] for details). The corresponding analysis was done recently in [3] using the value $F^{\rm WA}/F^{\rm peng} \simeq -0.3\ \xi_u/\xi_t$.

TABLE I. Results for the weak-annihilation form factors F_A, F_V and G_V for the $B^- \to \rho^-\gamma$ decay. The accuracy of our estimates is about 10%. The sum rule results are recalculated from [6] according to the relation $F_{A,V} = -F^{\rm SR}_{1,2} M_B/f_{\rho^-}$. The results from [7] are recalculated according to $F_{A,V} = -\frac{1}{2} f^{[7]}_{A,V}$ for $\bar\Lambda = 0.5$ GeV.

	Disp Approach	SR [6]	pQCD [7]
$-F_A$	0.120	0.073	>0.09
$-F_V$	0.092	0.091	>0.09
$\lvert G_V\rvert$	<0.004		

VI. CONCLUSION

We have analyzed the weak annihilation for the radiative decay $B \to \rho\gamma$ in the factorization approximation.
1. We have calculated the form factors F_A and F_V describing the photon emission from the B meson loop within the relativistic dispersion approach. We have performed the $1/m_b$ expansion of the form factors and demonstrated that the form factors of the dispersion approach exhibit a behaviour in agreement with the large-energy limit of QCD.
2. We have analyzed the contribution to the weak annihilation amplitude from the diagram when the photon is emitted from the loop containing only light quarks. For the parity-conserving process this quantity is related to the divergence of the axial-vector current

$$\langle \gamma(q_1)\rho^-(q_2)|\partial_\nu \bar{d}\gamma_\nu\gamma_5 u|0\rangle = e\epsilon_{q_1\epsilon_1^* q_2\epsilon_2^*}(Q_u+Q_d)\ \kappa\ f_\rho M_\rho/M_B^2. \tag{39}$$

with κ staying finite in the chiral limit $m \to 0$. This result means the violation of the classical equations of motion and represents an anomaly which has the same origin as the anomaly of the matrix element $\langle\gamma(q_1)\gamma(q_2)|\partial_\mu A_\mu|0\rangle$. The value of κ is found to be sensitive to subtle details of the ρ-meson wave function. Conservatively, we estimate $|\kappa| \lesssim 1$.
3. We have also included contact terms which were missed in some of the previous analyses. Numerical estimates for the weak annihilation contribution to the $B^- \to \rho^-\gamma$ amplitude are given in Eq. (38).

It was noticed in [5] that the weak annihilation mechanism is crucial for the $D \to \rho\gamma$ decays in which it dominates over the penguin mechanism. It is therefore very important to take into account all the contribitions to the weak annihilation amplitude listed above for a proper description of the rare radiative D decays.

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VII. APPENDIX: TRANSVERSITY OF THE PARITY-VIOLATING AMPLITUDE.

The parity-violating amplitude (11) is given by the sum of the two terms

$$A^{\rm PV}(B^- \to \rho\gamma^-) \sim A_1^{\rm PV} + A_2^{\rm PV} \tag{40}$$

where $A_1^{\rm PV} = \langle\rho^-(q_2)|\bar{d}\gamma_\nu u|0\rangle\langle\gamma(q_1)|\bar{u}\gamma_\nu\gamma_5 b|B^-(p)\rangle$ and $A_2^{\rm PV} = \langle\rho^-(q_2)\gamma(q_1)|\bar{d}\gamma_\nu u|0\rangle\langle 0|\bar{u}\gamma_\nu\gamma_5 b|B^-(p)\rangle$.
1. We start with $A_1^{\rm PV}$. Let us write $\langle\gamma(q_1)|\bar{u}\gamma_\nu\gamma_5 b|B^-(p)\rangle = e\,\epsilon_1^{*\mu} T^B_{\mu\nu}$ with

$$T^B_{\mu\nu}(p,q) = i\int dx e^{iqx}\langle 0|T(J_\mu(x), \bar{u}\gamma_\nu\gamma_5 b)|B^-(p)\rangle, \tag{41}$$

where $J_\mu(x) = \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d - \frac{1}{3}\bar{b}\gamma_\mu b$ is the electromagnetic quark current. The amplitude has the following Lorentz structure

$$T^B_{\mu\nu} = i\left(g_{\mu\nu}pq_1 - p_\mu q_{1\nu}\right)F_{1A}(q_1^2) + i\left(q_1^2\, p_\mu - pq_1\, q_{1\mu}\right)q_{1\nu}F_{2A}(q_1^2) + i\left(q_1^2\, p_\mu - pq_1\, q_{1\mu}\right)p_\nu F_{3A}(q_1^2) + \frac{ip_\mu p_\nu}{pq_1}C, \tag{42}$$

where C is the contact term. The contact term can be determined using the conservation of the electromagnetic current $\partial_\mu J_\mu = 0$, which leads to the relation

$$q_\mu T^B_{\mu\nu}(p,q) = -\langle 0|[\hat{Q}, \bar{d}\gamma_\nu\gamma_5 u]|B^-(p)\rangle = Q_B f_B p_\nu = -if_B p_\nu \tag{43}$$

for the B^- meson. This gives $C = -f_B$. Notice that for the B^0-meson the contact term is absent. For the radiative decay $q_1^2 = 0$ so we find

$$\begin{aligned} A_1^{\rm PV} &= ie\, f_\rho M_\rho \epsilon_1^{*\mu}\epsilon_2^{*\nu}\left\{(g_{\mu\nu}pq_1 - p_\mu q_{1\nu})F_{1A}(0) - p_\mu p_\nu\frac{2f_B}{M_B^2 - M_\rho^2}\right\} \\ &= ie\, f_\rho M_\rho \epsilon_1^{*\mu}\epsilon_2^{*\nu}\left\{(g_{\mu\nu}pq_1 - p_\mu q_{1\nu})F_{1A}(0) - p_\mu q_{1\nu}\frac{2f_B}{M_B^2 - M_\rho^2}\right\}. \end{aligned} \tag{44}$$

2. Using the equation of motion for the quark fields ($Q_u = 2/3\, e$, $Q_d = -1/3\, e$)

$$\begin{aligned} &i\gamma_\nu\partial^\nu q(x) = mq(x) - Q_q A_\nu\gamma^\nu q(x), \\ &i\partial^\nu\bar{q}(x)\gamma_\nu = -m\bar{q}(x) + Q_q A_\nu\bar{q}(x)\gamma^\nu, \end{aligned} \tag{45}$$

one obtains for $A_2^{\rm PV}$

$$\begin{aligned} A_2^{\rm PV} &= ip_\nu f_B\langle\rho^-\gamma|\bar{d}\gamma_\nu u|0\rangle = f_B\langle\rho^-\gamma|\partial_\nu(\bar{d}\gamma_\nu u)|0\rangle \\ &= -(Q_d - Q_u)e\,\langle\rho^-\gamma|A^\nu\bar{d}\gamma_\nu u|0\rangle = e\,\epsilon_1^{*\mu}\epsilon_2^{*\nu}g_{\mu\nu}f_\rho M_\rho f_B + O(m_u, m_d). \end{aligned} \tag{46}$$

3. For the sum we find

$$A_1^{\rm PV} + A_2^{\rm PV} = ie\, f_\rho f_B M_\rho\epsilon_1^{*\mu}\epsilon_2^{*\nu}\left(g_{\mu\nu}pq_1 - p_\mu q_{1\nu}\right)\left[F_{1A}(0) + \frac{2f_B}{M_B^2 - M_\rho^2}\right] \tag{47}$$

and obtain the relation (14) after setting $F_{1A} = 2F_A/M_B$. For the $B^0 \to \rho^0\gamma$ decay the term proportional to f_B is absent.